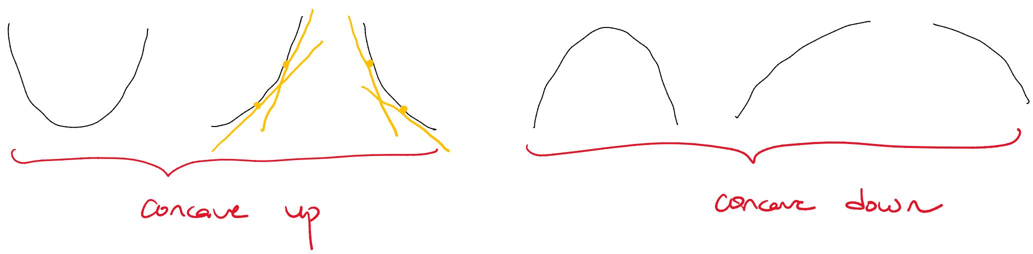
Section 4.5

Saturday, March 28, 2020 2:56 PM

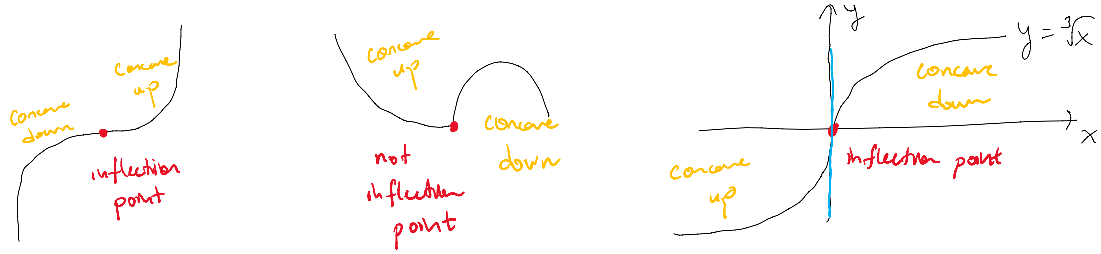
Defn: Let f be a differentiable function on an interval (a,b). We say that

- f is concave up on (a,b) if f' is increasing on (a,b).
- ______ ______ down ______ ______ decreasing on (a,b).



Defn: Let f be differentiable on (a,b). We say that f has an inflection point at $x=c$, so the inflection point is $(c, f(c))$, if

- the graph $y=f(x)$ admits a tangent line at $x=c$ and
- the concavities of f on opposite sides of c are different.



Theorem: Let f be differentiable on (a,b) and c be in (a,b). Then

a) If $f''(x) > 0$ on (a,b), then f is concave up on (a,b).

b) If $f''(x) < 0$ on (a,b), ______________________ down on (a,b).

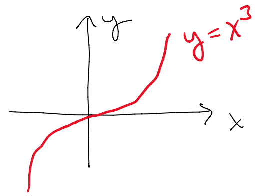
c) If f has an inflection point at $x=c$ and $f''(c)$ exists, then $f''(c)=0$.

Proof of a: Suppose that $f''(x)>0$ on (a,b). Then, since $(f')'(x) > 0$ on (a,b), f' is increasing on (a,b), and so, f is concave up on (a,b).

Proof of c: Suppose that f has an inflection point at $x=c$ and $f''(c)$ exists. Then, f' has a maximum or minimum at $x=c$ and $(f')'(c)$ exists. So $f''(c)=0$.

WARNING:

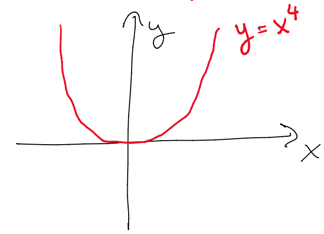
- f has an extreme value at c & $f'(c)$ exists $\Rightarrow f'(c)=0$ $\not\Leftarrow$



$f'(0)=0$

no local max or local min at $x=0$

- f has an inflection point at c and $f''(c)$ exists $\Rightarrow f''(c)=0$ $\not\Leftarrow$



$f''(0)=0$

no inflection point at $x=0$

Example: Find the intervals of increase and decrease, the extreme values and the concavity of

$$f(x)=3x^5-5x^3$$

Solution: We have that

$$f'(x)=15x^4-15x^2=15x^2(x^2-1)=15x^2(x-1)(x+1)$$

$$f''(x)=60x^3-30x=30x(2x^2-1)=60x\left(x-\frac{1}{\sqrt{2}}\right)\left(x+\frac{1}{\sqrt{2}}\right)$$

So $f'(x)=0 \Rightarrow x=0,\ 1$ or -1

$f''(x)=0 \Rightarrow x=0,\ \frac{1}{\sqrt{2}},\ \frac{-1}{\sqrt{2}}$

x	$-\infty$	-1		$\frac{-1}{\sqrt{2}}$		0		$\frac{1}{\sqrt{2}}$		1		$+\infty$
$f'(x)$	+	0	−		−	0	−		−	0	+	
$f''(x)$	−		−	0	+	0	−	0	+		+	
$f(x)$	↗ ⌢	local max	↘ ⌢	inflection point	↘ ∪	inflection point	↘ ⌢	inflection point	↘ ∪	local min	↗ ∪	

f is increasing on $(-\infty,-1)$ and $(1,+\infty)$

f is decreasing on $(-1,1)$

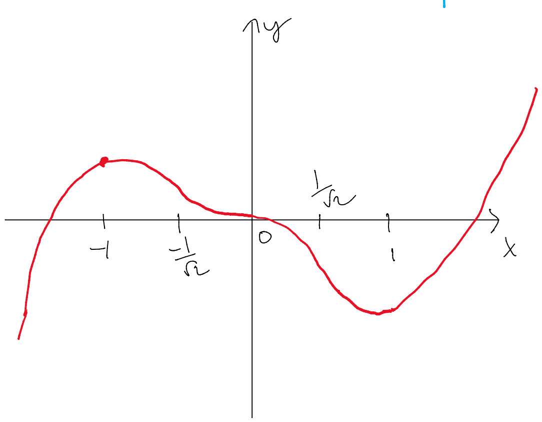
f has a local max at $x=-1$

______ min at $x=1$

f is concave up on $\left(\frac{-1}{\sqrt{2}},0\right)$ and $\left(\frac{1}{\sqrt{2}},+\infty\right)$

f is concave down on $\left(-\infty,\frac{-1}{\sqrt{2}}\right)$ and $\left(0,\frac{1}{\sqrt{2}}\right)$.

$\lim_{x\to\infty} f(x)=+\infty$ and $\lim_{x\to-\infty} f(x)=-\infty$ so there are no absolute extreme values.



The second derivative test:

a) If $f'(c)=0$ and $f''(c)>0$, then f has a local minimum at $x=c$.

b) ______ $f''(c)<0$, ______ maximum at $x=c$.

c) ______ $f''(c)=0$, ______ the test fails, f may have a local max, min or none.

Proof of a: Suppose that $f'(c)=0$ and $f''(c)>0$. Then we have that

$$\lim_{h\to 0}\frac{f'(c+h)}{h}=\lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h}=f''(c)>0$$

and so

$f'(c+h)>0$ for all sufficiently small $0<h$ and $f'(c+h)<0$ for all sufficiently small $h<0$. It follows from the first derivative test that f has a local minimum at c.

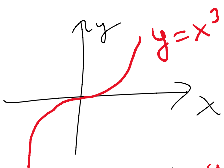
Proof of c: Consider the maps $f(x)=x^3$, $g(x)=x^4$, $h(x)=-x^4$. Then

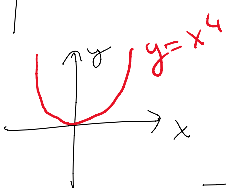
$f'(0)=f''(0)=g'(0)=g''(0)=h'(0)=h''(0)=0$ but

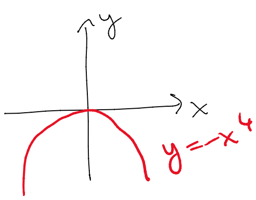
f does not have a local max or min at $x=0$.

g does ______ min at $x=0$.

h ______ max at $x=0$.







Example: Let $f(x)=(x^2-3).e^{-x}$ be defined on $[-2,6]$. Find the extreme values of f on $[-2,6]$. Roughly sketch the graph of f.

Solution: We have $f'(x)=2x.e^{-x}+(x^2-3).e^{-x}.(-1)=e^{-x}(-x^2+2x+3)$
$=e^{-x}.(-1).(x-3)(x+1)$ and

$$f''(x)=(2.e^{-x}+2x.e^{-x}.(-1))+(2x.e^{-x}.(-1)+(x^2-3).e^{-x})$$

$$=e^{-x}(2-2x-2x+x^2-3)=e^{-x}(x^2-4x-1)=e^{-x}(x-(2-\sqrt{5}))(x+(2-\sqrt{5}))$$

So $f'(x) = 0 \Rightarrow x = 3$ or -1

$$f''(x) = 0 \Rightarrow x = \frac{4 \mp \sqrt{16 - 4\cdot(-1)}}{2} = 2 \mp \sqrt{5}$$

x	-2		-1		$2-\sqrt{5}$		3		$2+\sqrt{5}$		6
f'		$-$	0	$+$		$+$	0	$-$		$-$	
f''		$+$		$+$	0	$-$		$-$	0	$+$	
f		$\searrow \cup$		$\nearrow \cup$		$\nearrow \cap$		$\searrow \cap$		$\searrow \cup$	
	local max		local min		inflection point		local max		inflection point		local min

- $f''(3) = e^{-3}.(9 - 12 - 1) = e^{-3}.(-4) < 0$

  So by the second derivative test f has a local max at $x = 3$

- $f''(-1) = e^{1}.(1 + 4 - 1) = e.4 > 0$. So f has a local min at $x = -1$ by the second derivative test

  (Observe that we also get this info from the table using the first derivative test.)